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AN ANALYTICAL STUDY OF GROUNDWATER FLUCTUATIONS IN UNCONFINED LEAKY AQUIFERS INDUCED BY MULTIPLE LOCALIZED RECHARGE AND WITHDRAWAL

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ABSTRACT
Commonly used analytical methods for assessing the effects of recharge and withdrawal on the groundwater flow system are based on an idealistic assumption that the aquifer's base is fully impervious. In reality, the hydrostratigraphic conditions are often complex and involve leakage induced flow between aquifer and the confining layers. In this study, a simple analytical procedure is presented for determining the spatial and temporal distribution of water head in an unconfined aquifer system due to multiple localized recharge and withdrawal at time-varying rates. A new transient function is introduced that can conveniently approximate the rising and recession limbs of any single recharge hydrograph. Solution of linearized two-dimensional groundwater flow equation under Dirichlet and Neumann boundary conditions is obtained using finite Fourier cosine transform with analytic inversion. The study has at least one clear advantage over the existing solutions that it accounts for the vertical leakage in water table buildup and drawdown analysis. A computational example demonstrates that the leakage induced flow plays an important role in recharge and withdrawal processes of unconfined aquifer system. The model results can be used for estimating aquifer's hydraulic properties and validation of numerical models.

KEYWORDS*:* Groundwater; Leakage; Recharge; Drawdown; Wells; Fourier cosine transform.

1. INTRODUCTION

The ever increasing demand of water for agriculture, industry and urban use often leads to overexploitation of groundwater resources. Sustainability and the efficiency of an aquifer mainly depend on the applied resource management practices. Special attention should be paid to the rational management of aquifers adjacent to aquatic ecosystems in ecologically sensitive regions where the interaction between surface and groundwater is critical. Furthermore, in coastal areas, excessive pumping of aquifers increases the risk of salinization and deteriorates water supplies both in quantity and quality. A number of pumping or injection wells and recharge basins can be irregularly distributed in space and operated intermittently in an aquifer system. Accurate prediction of water table in an aquifer that is subject to the combined action of pumping and recharge is a prerequisite for proper groundwater management. Therefore, there is a need for developing efficient mathematical model that can describes both spatial and temporal distribution of the groundwater head and the capture zones associated with water sources or withdrawals.

There are numerous mathematical models that have been presented in the literature to predict the groundwater response to the constant or periodically applied recharge in an unconfined aquifer (Hantush, 1967; Marino, 1974b; 1975; Latinopoulos, 1984; Manglik *et al.* 1997; Rai and Manglik 1999; Teloglou *et al.,* 2008; Bansal and Das, 2011; Manglik *et al.,* 2013). Rastogi and Pandey (1998) used a numerical model to simulate the groundwater head distribution in response to constant recharge from recharge basins of different shapes but equal areas. They showed that the groundwater mound underneath a basin was higher when its perimeter decreased.

Several investigators developed models to simulate the groundwater table response to recharge and pumping operations (Manglik *et al.,* 2004; Rai *et al.,* 2006). Chang and Yeh (2007) presented a mathematical model for simulation of groundwater flow in a homogeneous, anisotropic and sloping unconfined aquifer with transient recharge and multiple injection and/or extraction wells. Loáiciga (2008) presented closed-form solutions to the groundwater flow equation in marine island aquifers subject to time-independent and spatially variable dependent recharge and groundwater pumping. Xie *et al.* (2010) obtained analytical solutions to flow problems with discontinuous boundary conditions due to a circular source (i.e. pumping or recharging well). A comprehensive review of analytical and numerical techniques to solve well-hydraulic problems is presented by Yeh and Chang (2013).

Natural or artificial recharge and pumping rates are some of the most important variables for a regional groundwater model. A corollary of aquifer replenishment is the process of infiltration by which water moves vertically downward into the soil. Infiltration rate generally depends on soil properties such as texture, structure and the existence of layers. Furthermore, the sediment deposition at the bottom of the recharge basins gradually reduces the infiltration rate by clogging the soil pores. In three artificial recharge projects, Mousavi and Rezai (1999) evaluated the improvement of infiltration rate by scraping away various amounts of the upper soil materials. When removing the deposited sediment layer plus 15 cm of topsoil, maximum restoration degree of the initial infiltration capacity was observed. Mathematically, reduction of the infiltration rate for a single recharge period has been approximated by an exponentially decaying function (Ramana *et al.,* 1995; Rai and Singh, 1996; Teloglou *et al.,* 1997; Chang and Yeh, 2007; Singh and Jaiswal, 2010; Bansal, 2012). For multiple periods of recharge, the infiltration rate has been expressed as a series of line segments (Manglik *et al.,* 1997; Rai *et al.,* 2006; Rai and Manglik, 2012). Teloglou *et al.* (2008) introduced a generalized polynomial function, approximating the recharge rate during repeated cycles of recharge.

Apart from the recharge and pumping activities, leakage from a semipervious layer at the top or the bottom of an aquifer is a critical issue in analyzing regional groundwater balance. One of the restrictive assumptions in the aforementioned two-dimensional models is that the influence of leakage on the recharge or pumping-induced drawdown is not taken into account by considering an impermeable layer at the bottom of the aquifer. In natural systems, leaky beds occur far more often than perfectly impervious confining beds. Aquifers in deep sedimentary basins are part of multi-layered formations whose confining layers are often leaky. When a well in a leaky aquifer is pumped, water is not only withdrawn from the aquifer but also from the underlying aquitard (Malama *et al.,* 2007, Zlotnik and Tartakovsky, 2008). Similarly, when a leaky aquifer is replenished, a significant volume of water flows out through the aquifer-aquitard interface (Teloglou and Bansal, 2012).

In this paper, a new two-dimensional analytical solution is presented for groundwater flow in response to transient recharge and intermittently constant pumping rates from randomly-located basins/wells in an unconfined aquifer resting on a semipervious layer. Transient recharge is approximated by a new exponential function which describes both the rising and the recession limbs of any single recharge hydrograph. By incorporating the leakage term into equation, the upward or downward flow through the semipervious layer is controlled by the local water table height. The linearized form of groundwater flow equation with boundary conditions of prescribed head (Dirichlet) and of zero flux (Neumann) is solved using finite Fourier cosine transform. Omitting certain parameters in the general form of solution leads to more simplified solutions that previously obtained by other researchers. Sensitivity analysis was carried out for various scenarios of recharge and withdrawal operations by changing the hydraulic conductivity of semipervious layer. Numerical results are illustrated in figures and appropriately discussed.

2. MATHEMATICAL FORMULATION

As shown in Figure 1, we consider an L-shaped finite aquifer system consisting of an unconfined aquifer underlain by a semipervious bed. The aquifer is in contact with a water body that maintains a constant water head h_0 along the coastlines $x = A$ and $y = B$. The other two boundaries $x = 0$ and $y = 0$ of the aquifer are impervious, and thus, no flow condition is imposed across these boundaries. It is assumed that the system is in hydraulic equilibrium at the initial stages ($t = 0$) and h_0 is the initial elevation of the water table. Fluctuations in the phreatic surface are induced by multiple recharge

and withdrawal activities in the model domain. The recharge basins are rectangular shaped and the dimensions of injection and extraction wells are small compared to the dimension of the aquifer.
The two-dimensional groundwater flow in an unconfined aquifer with semipervious base is governed by the following nonlinear partial differential equation:

$$\frac{\partial}{\partial x}\left(h\frac{\partial h}{\partial x}\right)+\frac{\partial}{\partial y}\left(h\frac{\partial h}{\partial y}\right)+\frac{1}{K}P(x,y,t)=\frac{S}{K}\frac{\partial h}{\partial t}+\frac{k'}{K\,b'}(h-h_0) \quad (1)$$

where $h(x, y, t)$ is the water table height measured in the vertical direction from the semipervious base; K and S respectively denote the hydraulic conductivity and specific yield of the unconfined aquifer; k' and b' respectively denote the hydraulic conductivity and thickness of the semipervious bed. The term $P(x, y, t)$ in the left-hand side of the above equation simulates the combined effects of recharge and withdrawal activities in the model domain. These activities are carried out simultaneously using (i) rectangular basins of varying dimensions a_i x b_i with arbitrary spatial locations that are responsible for localized transient recharge, and (ii) wells of relatively lesser dimensions that are responsible for injection and/or extraction of the groundwater at time-varying rate. In other words

$$P(x,y,t)=\left[\sum_{i=1}^{p_1}R_i(x,y,t)+\sum_{j=1}^{p_2}\omega_j\,Q_j(t)\,\delta(x-x_j)\,\delta(y-y_j)\right] \quad (2)$$

Here, p_1 and p_2 denote the number of rectangular basin and wells respectively. The term ω_j is 1 or –1 according as the j^{th} well corresponds to an injection or extraction activity. δ is the Dirac delta function; $Q_j(t)$ is the transient rate of injection/extraction in the j^{th} well (j = 1, 2,…., p_2) centered at (x_j, y_j); and $R_i(x, y, t)$ is the transient recharge rate in the i^{th} basin (i = 1, 2, …, p_1) extending from $x_i \le x \le x_i + a_i$; $y_i \le y \le y_i + b_i$. If $f_i(t)$ denotes the rate of recharge in the i^{th} basin, then

$$R_i(x,y,t)=\begin{cases} f_i(t), & x_i\le x\le x_i+a_i\,;\; y_i\le y\le y_i+b_i \\ 0, & \text{otherwise}\end{cases} \quad (3)$$

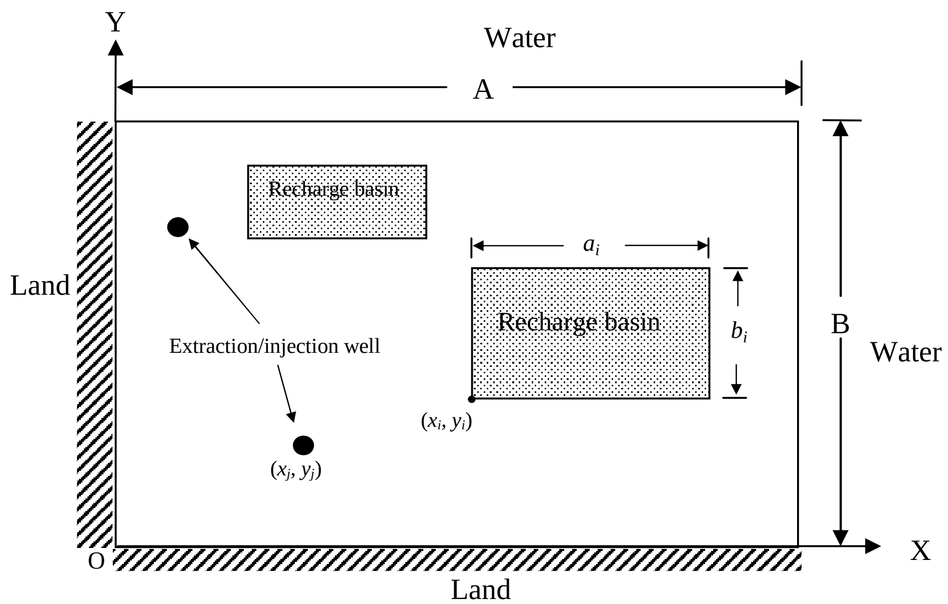


Figure 1. Overview of a leaky unconfined aquifer with multiple recharge basins and extraction wells

The initial and the boundary conditions are prescribed as follows:

$$h(x,y,t=0)=h_0 \quad (4a)$$

$$\left.\frac{\partial h}{\partial x}\right|_{x=0}=\left.\frac{\partial h}{\partial y}\right|_{y=0}=0 \quad (4b)$$

$$h(x,\;y=A,\;t)=h_0=h(x,\;y=B,\;t) \quad (4c)$$

Equation (1), due to its nonlinearity, does not admit an exact solution. Thus, linearization of this equation is inevitable for obtaining the closed form analytical solution. There are several methods to linearize the groundwater flow equation. Baumann (1952) suggested a linearization method in which the water head *h*(*x, t*) is replaced by the sum of characteristic (saturated) depth *D* and *ξ*(*x, t*), where *ξ*(*x, t*) is small compared with *D*. Werner (1957) linearized the Boussinesq's equation in terms of h^2 by replacing the first term on the right-hand side (RHS) of Eq.(1) as $(S/K\bar{h})\partial(h^2/2)/\partial t$. The parameters D and $\bar{h}$ that appear in both of the linearization methods denote the mean depth of saturation. Brutsaert (1994) preferred a linearization by which the term h associated with ∂h/∂*x* is replaced as *pD*, where *p* is a linearization constant (0 < *p* ≤ 1) and *D* is the saturated thickness of the aquifer.

In this study, we adopt the Werner's method and the value of $\bar{h}$ is obtained by successive use of the formula $\bar{h} = (h_0 + h_t)/2$, where h_0 is the initial water head and h_t is the water head at the current moment. This approach was suggested by Marino (1973, 1974a) and used in several analytical studies, such as Zissis *et al.* (2001), Teloglou *et al.* (2008), Teloglou and Bansal (2012), Bansal (2012) etc. The initial approximation of $\bar{h}$ is taken as h_0. It must be noted that the analytical solution of the linearized equation offer good approximation of the actual solution if the water table in the aquifer is subjected to small variations.

According to Werner's linearization method, equation (1) is written as

$$\frac{\partial^2 h^2}{\partial x^2} + \frac{\partial^2 h^2}{\partial y^2} + \frac{2}{K}P(x,y,t) = \frac{S}{K\hbar}\left(\frac{\partial h^2}{\partial t}\right) + \frac{k'}{K b' \hbar}(h^2 - h_0{}^2) \tag{5}$$

Setting $H(x, y, t) = h^2 - h_0{}^2$, we get

$$\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} + \frac{2}{K}P(x,y,t) = \frac{S}{K\hbar}\frac{\partial H}{\partial t} + \frac{k'}{K b' \hbar}H \tag{6}$$

The initial and boundary conditions reduce to

$$H(x,y,0) = 0 \tag{7a}$$

$$\left.\frac{\partial H}{\partial x}\right|_{x=0} = \left.\frac{\partial H}{\partial y}\right|_{y=0} = 0 \tag{7b}$$

$$H(x = A,\ y,\ t) = 0 = H(x,\ y = B,\ t) \tag{7c}$$

Equation (6) along with the conditions (7a) – (7c) can be solved using finite Fourier cosine transform. Define (Sneddon, 1974)

$$\xi(m,n,t) = F_{cc}\{H(x,y,t);(x,y)\to(m,n)\} = \int_{x=0}^{A}\int_{y=0}^{B} H(x,y,t)\cos\left(\frac{(2m+1)\pi x}{2A}\right)\cos\left(\frac{(2n+1)\pi y}{2B}\right)dx\,dy \tag{8}$$

The finite Fourier cosine transform reduces the equation (6) to the following form

$$-\beta_m{}^2\xi - \gamma_n{}^2\xi + \frac{2}{K}\bar{P}(m,n,t) = \frac{1}{\nu}\frac{d\xi}{dt} + c\,\xi \tag{9}$$

where

$$\beta_m = \frac{(2m+1)\pi}{2A};\ \gamma_n = \frac{(2n+1)\pi}{2B};\ c = \frac{k'}{K b' \hbar};\ \nu = \frac{K\hbar}{S} \tag{10}$$

$$\bar{P}(m,n,t) = \left[\sum_{i=1}^{p_1}\Omega_i\, f_i(t) + \sum_{j=1}^{p_2}\omega_j\,\eta_j\,Q_j(t)\right] \tag{11}$$

$$\Omega_i = \frac{1}{\beta_m\gamma_n}\left[\sin\{\beta_m(x_i + a_i)\} - \sin(\beta_m x_i)\right]\left[\sin\{\gamma_n(y_i + b_i)\} - \sin(\gamma_n y_i)\right] \tag{12}$$

and $$\eta_j = \cos(\beta_m x_j)\cos(\gamma_n y_j) \tag{13}$$

Rearranging equation (9) as

$$\frac{d\xi}{dt} + (\alpha + \nu c)\,\xi = \frac{2\nu}{K}\bar{P}(m,n,t) \tag{14}$$

where $\alpha = \nu\left(\beta_m^{\ 2} + \gamma_n^{\ 2}\right)$ (15)

Equation (14) can be solved using ordinary methods. Its solution is

$$\xi(m,n,t) = \frac{2\nu}{K} e^{-(\alpha+\nu c)t} \left[\sum_{i=1}^{p_1} \Omega_i \int_0^t e^{(\alpha+\nu c)\tau} f_i(\tau)\, d\tau + \sum_{j=1}^{p_2} \omega_j\, \eta_j \int_0^t e^{(\alpha+\nu c)\tau} Q_j(\tau)\, d\tau \right] \tag{16}$$

where τ is a variable of integration. $H(x, y, t)$ can now be obtained by inverting the finite Fourier transform as

$$H(x,y,t) = \frac{4}{AB} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \xi(m,n,t) \cos\left(\frac{(2m+1)\pi x}{2A} \right) \cos\left(\frac{(2n+1)\pi y}{2B} \right) \tag{17}$$

We obtain the solution of equation (5) as

$$h^2 = h_0^{\ 2} + \frac{8\nu}{ABK} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \cos\left(\beta_m x\right) \cos\left(\gamma_n y\right) e^{-(\alpha+\nu c)t} \times \left[\sum_{i=1}^{p_1} \Omega_i \int_0^t e^{(\alpha+\nu c)\tau} f_i(\tau)\, d\tau + \sum_{j=1}^{p_2} \omega_j\, \eta_j \int_0^t e^{(\alpha+\nu c)\tau} Q_j(\tau)\, d\tau \right] \tag{18}$$

The rate of recharge depends on several hydrologic parameters. For mathematical simplicity, some researchers (Rai and Singh, 1996; Chang and Yeh, 2007 etc.) used an exponentially decaying function of time to approximate the recharge rate. They assumed that

$$f_i(t) = P_i + N_i\, e^{-\lambda_i t} \tag{19}$$

where λ_i is a positive constant, determining the rate at which the recharge in the i^{th} basin reduces to a final value P_i from an initial value $P_i + N_i$. Similarly, the pumping rate in some studies (Zlotnik 2004; Chang and Yeh 2007 etc.) was considered constant for all values of time, i.e.

$$Q_j(t) = Q_j \tag{20}$$

Under such assumption, Equation (18) becomes

$$h^2 = h_0^{\ 2} + \frac{8\nu}{ABK} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \cos\left(\beta_m x\right) \cos\left(\gamma_n y\right) \times \left[\sum_{i=1}^{p_1} \Omega_i \left\{ \frac{P_i\left(1 - e^{-(\alpha+\nu c)t}\right)}{\alpha + \nu c} + \frac{N_i\left(e^{-\lambda_i t} - e^{-(\alpha+\nu c)t}\right)}{\alpha + \nu c - \lambda_i} \right\} + \sum_{j=1}^{p_2} \omega_j\, \eta_j\, Q_j \left(\frac{1 - e^{-(\alpha+\nu c)t}}{\alpha + \nu c} \right) \right] \tag{21}$$

If the model domain contains only a single rectangular basin (i.e. $p_1 = 1$, $p_2 = 0$) and the base of the unconfined aquifer is fully impervious ($c = 0$), then equation (21) reduces to the following form:

$$h^2 = h_0^{\ 2} + \frac{8\nu}{ABK} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \cos\left(\beta_m x\right) \cos\left(\gamma_n y\right) \left[\Omega_1 \left\{ \frac{P_1\left(1 - e^{-\alpha t}\right)}{\beta_m^{\ 2} + \gamma_n^{\ 2}} + \frac{N_1\left(e^{-\lambda_1 t} - e^{-\alpha t}\right)}{\beta_m^{\ 2} + \gamma_n^{\ 2} - \lambda_1} \right\} \right] \tag{22}$$

where Ω_1 can be obtained from equation (12) by setting $i = 1$. Equation (22) is same as the equation (15) of Rai and Singh (1996).

In practice, the recharge and withdrawal activities are carried out in discontinuous phases of varying durations (henceforth referred as cycles), each of which is separated by a dry/resting period. In such cases, equation (22) may not satisfactorily approximate the distribution of the water head. Manglik *et al.* (1997) suggested a method in which a sequence of linear elements of varying slopes and intercepts is used for approximating the recharge and withdrawal rate. They defined

$$f_i(t) = \begin{cases} r_{ij}\, t + c_{ij} & t_j \le t \le t_{j+1} \\ r_{ik} t + c_{ik} & t \ge t_k \end{cases} \tag{23}$$

where r_{ij} and c_{ij} respectively denote the slope and length of intercept of the j^{th} element. One perceived drawback of this method is that the approximation of multiple cycles of recharge and withdrawal would need large number of line segments, and thus, the computational cost of the method would be very high. We propose a function that can closely approximate a complete individual cycle of recharge. The function is defined as

$$f_i(t) = q_i(t - r_i)e^{s_i t} \tag{24}$$

where q_i, r_i and s_i are constants. It is shown in Figure 2 that the rising and recession limbs of the recharge of a single hydrograph can be satisfactorily approximated by this function. Here, the solid and dotted lines denote the approximations using equation (24) and (23) respectively. The dry period corresponds to $q_i = 0$. A sequence of such functions can be used for approximation of the complete recharge operation consisting of multiple cycles of recharge of varying duration. If $[t_{ik}, t_{i(k+1)}]$ denotes the time interval of k^{th} recharge cycle of the i^{th} basin in which the recharge rate is given by

$$f_{ik}(t) = q_{ik}(t - r_{ik})e^{s_{ik} t}, \qquad k = 1, 2, ... \tag{25}$$

then we get

$$\int_0^t e^{(\alpha+\nu c)\tau} f_i(\tau)\, d\tau = \sum_{k=1}^{n_i - 1} D_{ik} + D_{i n_i} \tag{26}$$

where n_i is the current cycle in the i^{th} basin, t is the current time and $t_{i1} = 0$ for all i. While the first term in the RHS simulates the effects of first $n_i - 1$ recharge cycles which are already over; the last term signifies the contribution of the current cycle. The terms D_{ik} and D_{in_i} are given as follows:

$$D_{ik} = \frac{q_{ik}}{(s_{ik} + \alpha + \nu c)}\left[e^{(s_{ik}+\alpha+\nu c)t_{i(k+1)}} \left\{ (t_{i(k+1)} - r_{ik}) - \frac{1}{(s_{ik} + \alpha + \nu c)} \right\} - e^{(s_{ik}+\alpha+\nu c)t_{ik}} \left\{ (t_{ik} - r_{ik}) - \frac{1}{(s_{ik} + \alpha + \nu c)} \right\} \right] \tag{27a}$$

and

$$D_{in_i} = \frac{q_{in_i}}{(s_{in_i} + \alpha + \nu c)}\left[e^{(s_{in_i}+\alpha+\nu c)t} \left\{ (t - r_{in_i}) - \frac{1}{(s_{in_i} + \alpha + \nu c)} \right\} - e^{(s_{in_i}+\alpha+\nu c)t_{in_i}} \left\{ (t_{in_i} - r_{in_i}) - \frac{1}{(s_{in_i} + \alpha + \nu c)} \right\} \right] \tag{27b}$$

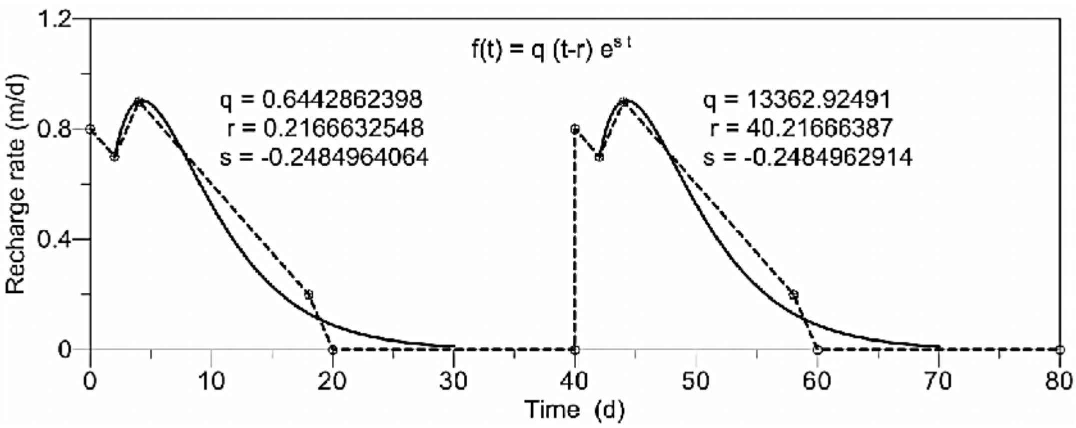


Figure 2. Approximation of rising and recession limbs of recharge hydrograph by linear elements and curve fitting by equation (24)

Similarly, the pumping operation in an injection/extraction well typically consists of discontinuous cycles of varying durations. Without much loss of generality, it can be assumed that the pumping rate throughout a particular cycle remains constant (Figure 3). A dry period corresponds to $Q_j = 0$. If $[t_{jl}, t_{j(l+1)}]$ denotes the time interval of l^{th} pumping cycle in the j^{th} well in which the pumping rate is given as

$$Q_{jl}(t) = Q_{jl}, \qquad l = 1, 2, ... \tag{28}$$

then

$$\int_0^t e^{(\alpha+\nu c)\tau} Q_j(\tau)\, d\tau = \sum_{l=1}^{m_j-1} P_{jl} + P_{j m_j} \tag{29}$$

where m_j is the current cycle in the j^{th} well, t is the current time and $t_{j1} = 0$ for all j. The terms P_{jl} and $P_{j m_j}$ are given by

$$P_{jl} = \frac{Q_{jl}}{(\alpha+\nu c)} \left[e^{(\alpha+\nu c) t_{j(l+1)}} - e^{(\alpha+\nu c) t_{jl}} \right] \tag{30a}$$

and

$$P_{j m_j} = \frac{Q_{j m_j}}{(\alpha+\nu c)} \left[e^{(\alpha+\nu c) t} - e^{(\alpha+\nu c) t_{j m_j}} \right] \tag{30b}$$

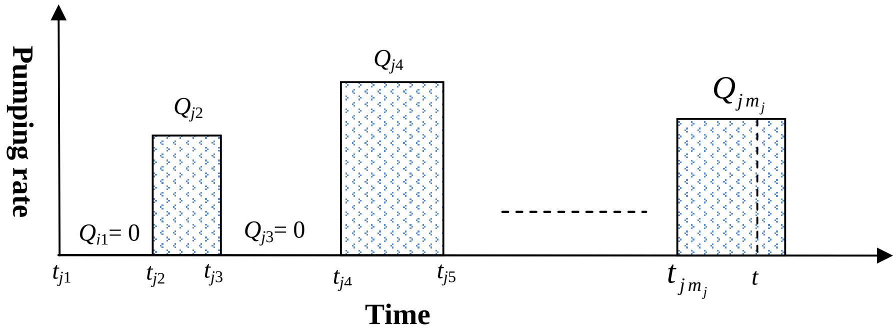


Figure 3. Complete pumping operation in the j^{th} well consisting of intermittent cycles of constant pumping

The water head distribution in the unconfined aquifer induced by multiple cycles of recharge and withdrawal can now be given as

$$h^2 = h_0^{\,2} + \frac{8\nu}{ABK} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \cos(\beta_m x) \cos(\gamma_n y) e^{-(\alpha+\nu c)t} \times \left[\sum_{i=1}^{p_1} \Omega_i \left\{ \sum_{k=1}^{n_i-1} D_{ik} + D_{i n_i} \right\} + \sum_{j=1}^{p_2} \omega_j \eta_j \left\{ \sum_{l=1}^{m_j-1} P_{jl} + P_{j m_j} \right\} \right] \tag{31}$$

where the terms D_{ik} (k = 1, 2, …, n_i) and P_{jl} (l = 1, 2, …, m_j) can be obtained from equations (27) and (30).

3. DISCUSSION OF RESULTS

To illustrate the combined effects of time-varying recharge, withdrawal and bed leakage on the water table fluctuations, we consider a leaky aquifer system of dimension 600 m × 400 m. The numerical values of controlling parameters are taken as: h_0 = 15 m, K = 10 m d^{-1}, S = 0.25 and b' = 1.5 m. Localized transient recharge is applied through two rectangular basins namely R-1 and R-2 centered at (150 m, 100 m) and (450 m, 300 m) respectively, each of dimension 50 m × 50 m. Moreover, intermittent extraction is considered from two wells W-1 and W-2 located at (150 m, 300 m) and (450 m, 100 m) respectively (Figure 4).

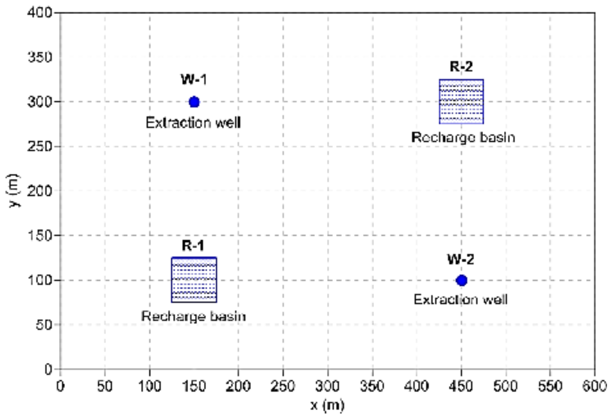


Figure 4. A plan view of aquifer with the relative positions of basins and wells in the numerical example

The dimensions of the extraction wells are small compared to that of the model domain. Recharge and withdrawal activities are taking place simultaneously in the form of disjoint cycles of varying rates and durations, which are separated by a resting period of an arbitrary duration. In the current example, we consider two distinct recharge schemes in basins R-1 and R-2. As shown in Figures 5 and 6, each scheme consists of two disjoint recharge cycles which are preceeded and followed by a dry spell. The rising and recession limbs of the recharge cycles are simulated using the transient function defined by equation (25). Values of parameters q_{ik}, r_{ik} and s_{ik} (i = 1, 2 and k = 1, 2, 3, 4, 5) used for approximation of the recharge rates are described in Table 1. Extraction activities in wells W-1 and W-2 are carried out in two intermittent cycles. As shown in Figs. 5 and 6, each cycle is preceded and followed by a resting period (i.e. $Q_{ik} = 0$). Values of pumping rates Q_{ik} for i = 1, 2 and k = 1, 2, 3, 4, 5 are presented in Table 2.

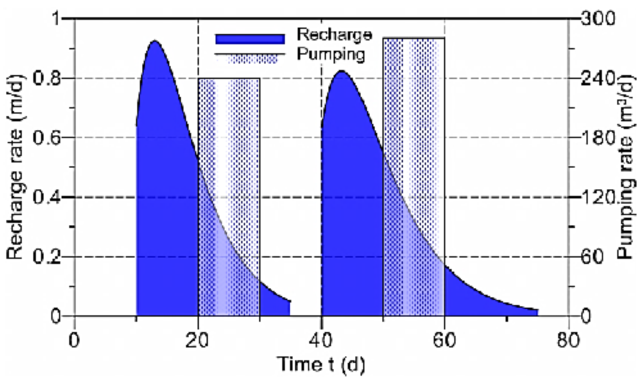


Figure 5. Recharge and pumping scheme in basin R-1 and well W-1

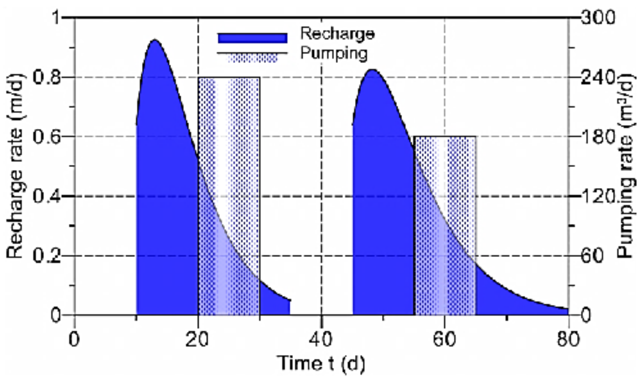


Figure 6. Recharge and pumping scheme in basin R-2 and well W-2

A FORTRAN programme is written to compute the water head distribution from equation (31). Average saturated depth of the aquifer is calculated using iterative formula $\bar{h} = \frac{(h_0 + h_t)}{2}$ where h_0 is the height of initial water table and h_t is the height at time t, at the end of which $\hbar$ is calculated (Marino, 1973; 1974a; b).

Table 1. Values of parameters q_{ik}, r_{ik} and s_{ik} for recharge schemes in basins R-1 and R-2

Cycle	Recharge Basin 1 (R-1)				Recharge Basin 2 (R-2)			
k	t (d)	q_{1k}	r_{1k}	s_{1k}	t (d)	q_{2k}	r_{2k}	s_{2k}
1	0 – 9	0	-	-	0 – 9	0	-	-
2	10 –35	3.02519	8.25375	– 0.21092	10 – 35	3.02519	8.25375	– 0.21092
3	36 –39	0	-	-	36 – 44	0	-	-
4	40 – 75	277.378	37.4956	– 0.17499	45 – 80	665.36183	42.47564	– 0.17499

From computational viewpoint, application of equation 31 is straightforward. The only minor disadvantage is that the sine, cosine and time exponent terms present in the right-hand side of equation (31) produces oscillatory and slow converging values of the double summation. In order to eliminate the oscillations completely and ensure the convergence of the results, one has to take large number of terms (m = 800 and n = 800) and dense grid spacing (5 m x 5 m). Numerical values of water head heights are obtained for three different values of hydraulic conductivity of the semipervious base, viz. k' = 0.25, 0.5 and 0.75 m d^{-1}. The corresponding values of hydraulic resistance b'/k' are 6, 3 and 2 d respectively.

Table 2. Extraction rate Q_{jk} in wells W-1 and W-2

Cycle	Extraction Well 1 (W-1)		Extraction Well 2 (W-2)	
k	t (d)	Q_{1k} ($m^3 d^{-1}$)	t (d)	Q_{2k} ($m^3 d^{-1}$)
1	0 – 19	0	0 – 19	0
2	20 – 30	240	20 – 30	240
3	31 – 49	0	31 – 54	0
4	50 – 60	280	55 – 65	180
5	61 – 75	0	66 – 75	0

Fluctuations in the water table beneath recharge basins are observed after 10 d when the first cycle of recharge has commenced in both R-1 and R-2. However, significant variations in the water table are observed only after t = 20 d when the extraction from W-1 and W-2 has started. To outline the combined effects of recharge and withdrawal, the transient profiles of water head along the line y = 300 m passing through the centers of W-1 and R-2 are plotted at t = 25 d in Fig. 7(a). At this stage, the pumping rate in W-1 is 240 $m^3 d^{-1}$ and the recharge rate in R-2 is approximately 0.26 m d^{-1}. Fig. 7(a) demonstrates that evolution of groundwater mound beneath recharge basins largely depends on the hydraulic resistance of the semipervious base. Groundwater mound develops around the vertical line passing through the centre of R-2 whose height increases with the hydraulic resistance. It is observed that the peak values of head gain $h - h_0$ at t = 25 d for b'/k' = 2, 3 and 6 d are respectively 0.337, 0.434 and 0.639 m. Aquifer systems whose base layer is of higher hydraulic resistance are lesser prone to downward leakage, and thus, exhibit higher level of water table in response to recharge. However, the variation might depend on several factors including aquifer parameters and hydraulic properties of the underlying aquitard. Depletion of water table induced by pumping from W-1 is also affected by the bed leakage. For instance, the drawdown $h_0 - h$ at t = 25 d for b'/k' = 2, 3 and 6d are 1.2, 1.26 and 1.35 m respectively. In leaky aquifers, the extraction from wells is partially supported by the aquifer; the balance of groundwater is supplied by leakage induced vertical flow from other hydraulically connected sources. Consequently, the lateral and vertical extent of the cone of depression is mitigated by the bed leakage. These observations establish the importance of leakage induced by the recharge and withdrawal mechanisms in an unconfined aquifer system lying on a semipervious base.

Fig. 7(b) presents the profiles of transient water head at t = 60 d when the second cycle of withdrawal in W-1 and also the second cycle of recharge in R-2 are in progress. Values of other controlling parameters are kept same. At this moment, the pumping rate in W-1 is 280 m^3 d^{-1} and recharge rate in R-2 is reduced to approximately 0.32 m/d from its maximum value of 0.79 m d^{-1}. One sees here that the mound height for b' /k' = 2, 3 and 6 d are respectively 0.413, 0.529 and 0.77m which are comparatively higher than the corresponding values at t = 25 d. To certain extent, the recharge water of the first cycle is responsible for the higher growth; however, the interplay between recharge rate and hydraulic properties of the underlying aquitard cannot be ignored. Groundwater depletion caused by pumping from W-1 is also affected by the first cycle of withdrawal. As a result, the drawdown levels viz. 1.41, 1.47 and 1.59 m for b' /k' = 2, 3 and 6 d are higher in the present case. Also, the lateral extent of depression cone is wider than that of t = 25d.

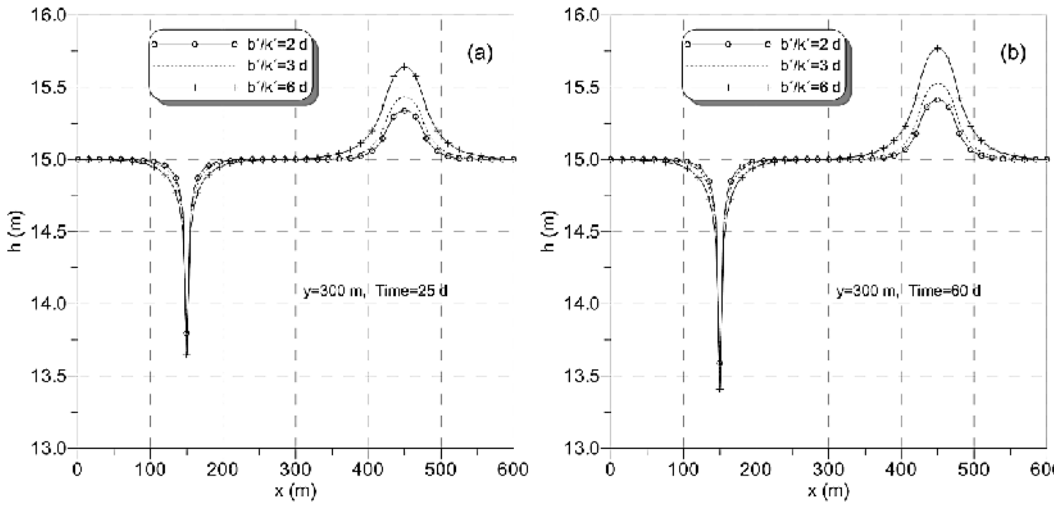
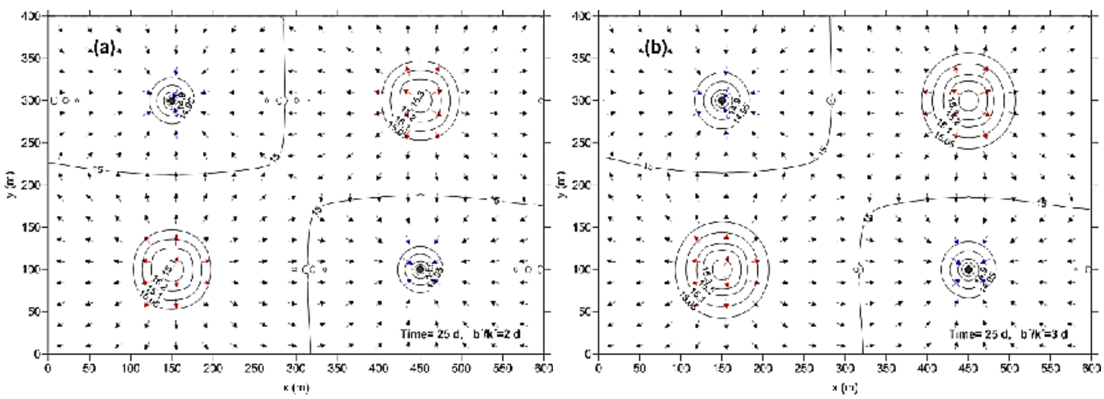


Figure 7. Water head distribution along the line y = 300 m for b' / k' = 2, 3 and 6 d at (a) 25 d, and (b) 60 d

Water table contours and hydraulic gradients diagrams at t = 25 and 60 d are plotted in Figures 8 and 9 respectively. The contours are symmetric about the centre lines of recharge basins and extraction wells.



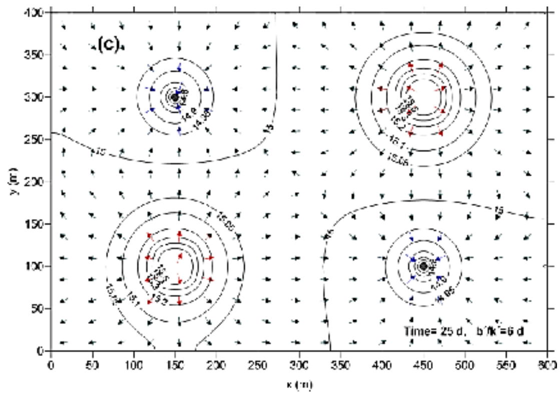


Figure 8. Water table contours and hydraulic gradients in the aquifer at t = 25 d for (a) b' / k' = 2d, (b) b' / k' = 3d and (c) b' / k' = 6d

Since the first cycle of recharge is identical in both R-1 and R-2; development of water table under these basins is almost similar. However, as indicated by Figures 8(a) – 8(c), the lateral and vertical expansion of groundwater mound is significantly controlled by the hydraulic resistance of semipervious bed. For example, the equipotential h = 15.05 for b'/k' = 2d barely reaches to the line y = 50, whereas it almost approaches to the no-flow boundary (y = 0) for higher hydraulic resistance b'/k' = 6d. Similarly, the fact that the pumping rate and duration of the first withdrawal cycle in W-1 and W-2 are identical, suggests that the drawdown due to extraction from W-1 and W-2 should follow the same pattern. This behavior can be clearly observed from Figure 8. As the recharge scenario changes after first cycle, water table under R-1 and R-2 develop differently (Fig. 9). The recharge rate in R-2 during the recession period preceding time t = 60 d is higher than that of R-1. As a result, the growth of water table beneath R-2 is comparative higher. Similarly, the cones of depression under W-1 and W-2 also differ in vertical and horizontal dimension due to varying rates of withdrawal.

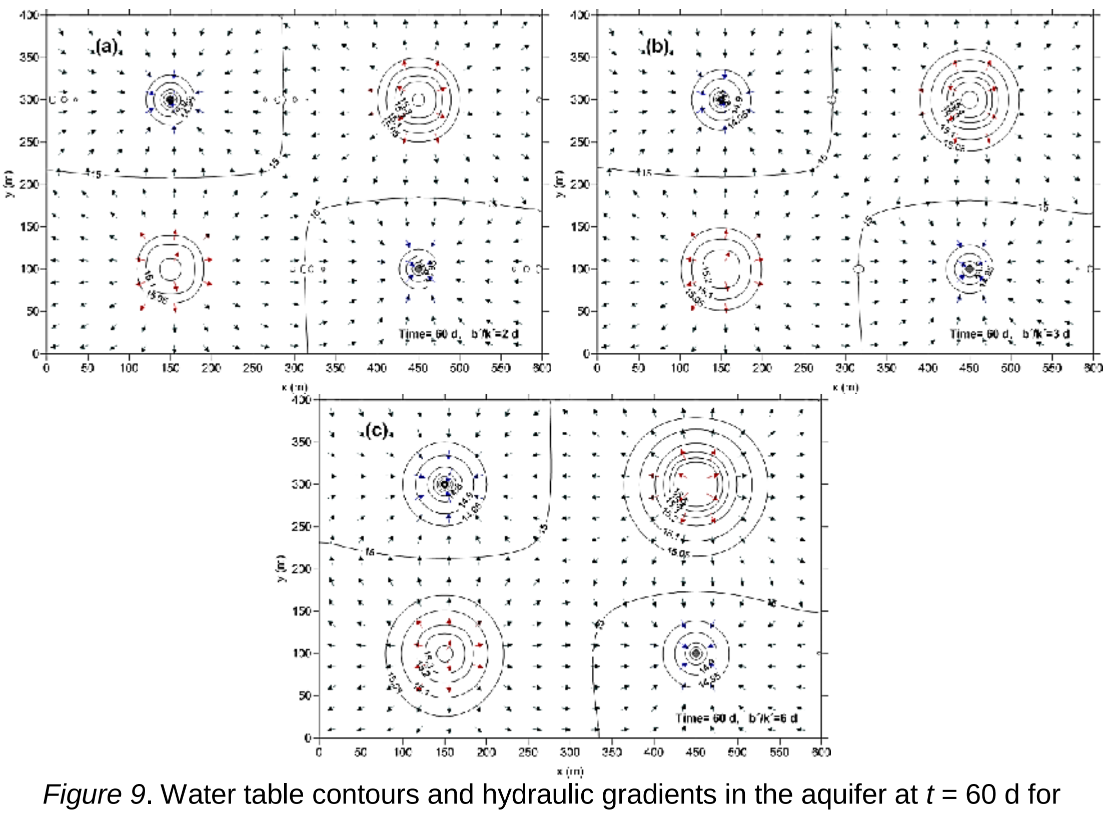


Figure 9. Water table contours and hydraulic gradients in the aquifer at t = 60 d for (a) b'/k' = 2d, (b) b'/k' = 3d and (c) b'/k' = 6d

4. CONCLUSIONS

Leakage can occur both in confined and unconfined aquifers. Existing solutions developed for assessing water table fluctuations in response to recharge and pumping activities, assume that the base of the aquifer is impervious. This restrictive approach treats the aquifer as an isolated unit in which recharge and withdrawal are assumed to be 100% linked with the groundwater. Such solutions may not be extendable to the natural systems consisting of leaky aquifers, e.g. multi-layered aquifer in deep sedimentary basins where the recharge and withdrawal mechanism of a layer is partially controlled by the hydrological properties of the underlying aquitard.

In this study, we develop an analytical procedure for assessing the effects of recharge and pumping induced leakage flow on the spatio-temporal distribution of water head. The unconfined leaky aquifer is considered to be homogeneous, and the variations in hydraulic conductivity with the spatial variables are neglected. Recharge and withdrawal rates are considered as function of space and time coordinates. Unlike previous studies in which the transient recharge is approximated by an exponentially decreasing function of time or by sequence of several linear elements of varying slopes and intercepts; a new function is proposed that can conveniently approximate the typical rising and recession limbs of any single recharge hydrograph. Closed form expressions are developed that can simulate the combined effects of multiple cycles of recharge, pumping operation and leakage on the water head distribution. In the limiting case, the analytical results reduce to some previously known results. It is demonstrated in the study that the pumping induced drawdown is partially supplied by the underlying aquitard. Similarly, a substantial volume of water flows out through the aquifer-aquitard interface when a leaky aquifer is replenished. As a result, the height of groundwater mound beneath recharge basins increases with the hydraulic resistance b'/k', and the lateral and vertical extent of the cone of depression due to pumping-induced drawdown is mitigated by the bed leakage. The presented solutions for water head distribution in leaky unconfined aquifers are of practical value from hydrological and geotechnical perspectives. The analytical solution could prove a very useful tool for simulation in the preliminary stage of a groundwater modeling study. Also, this model can be coupled with an optimization procedure for estimating aquifer hydraulic parameters at the regional scale. Apart from practical application, analytical models are useful for testing the accuracy of numerical schemes which are more effective in groundwater management of aquifers with complex hydrogeological settings and irregular boundaries.

NOMENCLATURE

Latin symbols

h	water table height above semipervious bed [L]
x, y	spatial coordinates [L]
t	time [T]
h_0	initial elevation of water head in the aquifer [L]
$\bar{h}$	mean saturated depth of the aquifer [L]
k'	hydraulic conductivity of the semipervious bed [LT^{-1}]
b'	thickness of the semipervious bed [L]
$a_i \times b_i$	dimension of the i^{th} rectangular recharge basins [L^2]
$f_i(t)$	rate of recharge at any time t in the i^{th} basin [LT^{-1}]
i, j, k, l	index parameters [-]
p_1, p_2	number of recharge basins and injection/extraction wells respectively [-]
q_{ik}, r_{ik}, s_{ik}	parameters controlling the k^{th} recharge cycle in the i^{th} basin [-]
K	hydraulic conductivity of the unconfined aquifer [LT^{-1}]
S	specific yield [-]
L	length of the aquifer [L]
B	width of the aquifer [L]
$P(x, y, t)$	variable recharge and withdrawal rate [LT^{-1}]
$R_i(x, y, t)$	transient recharge rate in the i^{th} basin [LT^{-1}]
Q_j	transient rate of injection/extraction at any time in the j^{th} well [$L^3 T^{-1}$]
Q_{jl}	transient rate of injection/extraction in l^{th} cycle of j^{th} well [$L^3 T^{-1}$]
H	$h^2 - h_0^2$ [L^2]

Greek symbols

ω_j	a variable which is 1 for an injection well and –1 for an extraction well [-]
δ	Dirac delta function [L^{-1}]

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